数理解析研究所講究録
第 1971 巻 2015 年 85-99

DECAY PROPERTIES OF SOLUTIONS TO THE STOKES EQUATIONS WITH SURFACE TENSION AND GRAVITY; ITS APPLICATION

HIROKAZU SAITO AND YOSHIHIRO SHIBATA

ABSTRACT. The aim of this article is to show the existence of a unique strong solution global in time for suitable initial data and large-time behavior of the solution for a free boundary problem of the incompressible Navier-Stokes equations in half-space-like domains. Our approach is based on the contraction mapping theorem combined with the maximal L_p-L_q regularity property of the linearized system and decay properties of solutions to the Stokes equations with surface tension and gravity.

1. INTRODUCTION

This article is a brief survey of [14] and [16], mainly.

1.1. **Problem.** We consider in this article the following free boundary problem of the incompressible Navier-Stokes equations in $\mathbf{R}^3$:

$$(1.1)\qquad \begin{cases} \rho\,(\partial_t\mathbf{v}+(\mathbf{v}\cdot\nabla)\mathbf{v}) = \operatorname{Div}\mathbf{S}(\mathbf{v},p)-\rho c_g\mathbf{e}_3 & \text{in } \Omega(t),\ t>0,\\ \operatorname{div}\mathbf{v}=0 & \text{in } \Omega(t),\ t>0,\\ \mathbf{S}(\mathbf{v},p)\mathbf{n}_\Gamma = c_\sigma\kappa_\Gamma\mathbf{n}_\Gamma & \text{on } \Gamma(t),\ t>0,\\ V_\Gamma=\mathbf{v}\cdot\mathbf{n}_\Gamma & \text{on } \Gamma(t),\ t>0,\\ \mathbf{v}|_{t=0}=\mathbf{v}_0 & \text{in } \Omega_0,\\ \Gamma|_{t=0}=\Gamma_0. \end{cases}$$

Here Γ_0 is a given initial surface defined by the graph of a scalar function $h_0=h_0(x')$ for $x'=(x_1,x_2)\in\mathbf{R}^2$, that is, $\Gamma_0=\{(x',x_3)\mid x'=(x_1,x_2)\in\mathbf{R}^2,\ x_3=h_0(x')\}$; $\Omega_0=\{(x',x_3)\mid x'\in\mathbf{R}^2,\ x_3<h_0(x')\}$ is the initial domain occupied by some Newtonian fluid with viscosity coefficient $\mu>0$; $\mathbf{v}_0=\mathbf{v}_0(x)=(v_{01}(x),v_{02}(x),v_{03}(x))^{T}$[1)] is a given initial velocity field of the fluid. The positive constants ρ, c_g, and c_σ describe the density, gravitational acceleration, and surface tension coefficient, respectively, and also $\mathbf{e}_3=(0,0,1)^T$.

Let $\Gamma(t)$ and $\Omega(t)$ be the position of Γ_0 and the region occupied by the fluid at time $t>0$, respectively. Note that both of them are unknown in the system (1.1). Furthermore, the unknowns $\mathbf{v}=\mathbf{v}(x,t)=(v_1(x,t),v_2(x,t),v_3(x,t))^T$ and $p=p(x,t)$ denote the velocity filed and the pressure field at $x\in\Omega(t)$ for $t>0$, respectively. The stress tensor $\mathbf{S}(\mathbf{v},p)$ is then given by

$$\mathbf{S}(\mathbf{v},p)=-p\mathbf{I}+\mu\mathbf{D}(\mathbf{v}),\quad \mathbf{D}(\mathbf{v})=\nabla\mathbf{v}+(\nabla\mathbf{v})^T=(\partial_iv_j+\partial_jv_i),$$

where $\mathbf{I}$ is the 3×3 identity matrix and $\partial_i=\partial/\partial x_i$ for $i=1,2,3$.

We set, for any 3×3 matrix $\mathbf{M}=(M_{ij})$ and 3-vector $\mathbf{v}=(v_1,v_2,v_3)$,

$$\operatorname{Div}\mathbf{M}=\left(\sum_{j=1}^3\partial_jM_{1j},\sum_{j=1}^3\partial_jM_{2j},\sum_{j=1}^3\partial_jM_{3j}\right)^T,\quad \operatorname{div}\mathbf{v}=\sum_{j=1}^3\partial_jv_j,$$

[1)] $\mathbf{M}^T$ describes the transposed $\mathbf{M}$.

$$(\mathbf{v}\cdot\nabla)\mathbf{v} = \left(\sum_{j=1}^{3} v_j\partial_j v_1, \sum_{j=1}^{3} v_j\partial_j v_2, \sum_{j=1}^{3} v_j\partial_j v_3\right)^T.$$

It then holds that

$$\text{the } i\text{th component of } \operatorname{Div}\mathbf{S}(\mathbf{v},p) = -\partial_i\pi + \mu(\Delta v_i + \partial_i \operatorname{div}\mathbf{v}).$$

We suppose that the unknown free surface $\Gamma(t)$ and domain $\Omega(t)$ are given by a scalar function $h = h(x',t)$ as follows:

$$\Gamma(t) = \{(x',x_3) \mid x' \in \mathbf{R}^2,\ x_3 = h(x',t)\}, \quad \Omega(t) = \{(x',x_3) \mid x' \in \mathbf{R}^2,\ x_3 < h(x',t)\}.$$

In addition, we denote the unit outward normal vector on $\Gamma(t)$ by $\mathbf{n}_\Gamma$, the evolution velocity of $\Gamma(t)$ with respect to $\mathbf{n}_\Gamma$ by V_Γ, and the mean curvature of $\Gamma(t)$ by κ_Γ, respectively. The unit outward normal vector on Γ_0 is analogously denoted by $\mathbf{n}_0$. It then holds, for $\nabla' h = (\partial_1 h, \partial_2 h)^T$ and $\Delta' h = \sum_{j=1}^2 \partial_j^2 h$, that

$$\mathbf{n}_\Gamma = \frac{1}{\sqrt{1+|\nabla' h(x',t)|^2}}\begin{pmatrix}-\nabla' h(x',t)\\ 1\end{pmatrix}, \quad V_\Gamma = \frac{\partial_t h(x',t)}{\sqrt{1+|\nabla' h(x',t)|^2}},$$

$$\kappa_\Gamma = \nabla'\cdot\left(\frac{\nabla' h(x',t)}{\sqrt{1+|\nabla' h(x',t)|^2}}\right) = \Delta' h - G_\kappa(h),$$

where

$$G_\kappa(h) = \frac{|\nabla' h(x',t)|^2\Delta' h(x',t)}{(1+\sqrt{1+|\nabla' h(x',t)|^2})\sqrt{1+|\nabla' h(x',t)|^2}} + \sum_{j,k=1}^{2}\frac{\partial_j h(x',t)\partial_k h(x',t)\partial_j\partial_k h(x',t)}{(1+|\nabla' h(x',t)|^2)^{3/2}}.$$

Set $\pi = p + \rho c_g x_3$ in (1.1), and we see, by $\mathbf{e}_3 = \nabla x_3$, that the system (1.1) are reduced to

$$\begin{cases} \rho(\partial_t\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v}) - \mu\Delta\mathbf{v} + \nabla\pi = 0 & \text{in } \Omega(t),\ t>0,\\ \operatorname{div}\mathbf{v} = 0 & \text{in } \Omega(t),\ t>0,\\ \mathbf{S}(\mathbf{v},\pi)\mathbf{n}_\Gamma + (\rho c_g h - c_\sigma\Delta' h)\mathbf{n}_\Gamma = -c_\sigma G_\kappa(h)\mathbf{n}_\Gamma & \text{on } \Gamma(t),\ t>0,\\ \partial_t h + \mathbf{v}'\cdot\nabla' h - \mathbf{v}\cdot\mathbf{e}_3 = 0 & \text{on } \Gamma(t),\ t>0,\\ \mathbf{v}|_{t=0} = \mathbf{v}_0 & \text{in } \Omega_0,\\ h|_{t=0} = h_0 & \text{on } \mathbf{R}^2, \end{cases} \tag{1.2}$$

where $\mathbf{v}'\cdot\nabla' h = \sum_{j=1}^2 v_j\partial_j h$.

1.2. **Reduction to a fixed domain problem.** The system (1.2) are reduced to a nonlinear problem on a fixed domain by the so-called *Hanzawa transformation*. To consider the transformation, we introduce the following auxiliary problem:

$$\begin{cases} \Delta H = 0 & \text{in } \mathbf{R}^3_-,\ t\ge 0,\\ H = h & \text{on } \mathbf{R}^3_0,\ t\ge 0, \end{cases} \tag{1.3}$$

where

$$\mathbf{R}^3_- = \{(x',x_3) \mid x'\in\mathbf{R}^2,\ x_3<0\}, \quad \mathbf{R}^3_0 = \{(x',x_3) \mid x'\in\mathbf{R}^2,\ x_3=0\}.$$

Let Θ be the transformation as follows:

$$\Theta : \mathbf{R}^3_- \times (0,\infty) \ni (\xi,\tau) \mapsto (x,t) \in \bigcup_{s\in(0,\infty)} \Omega(s) \times \{s\},$$
$$\Theta(\xi,\tau) = (\xi_1, \xi_2, \xi_3 + H(\xi,\tau), \tau).$$

We then define, for $f : \bigcup_{s\in(0,\infty)} \Omega(s) \times \{s\} \to \mathbf{R}$ and $g : \mathbf{R}^3_- \times (0,\infty) \to \mathbf{R}$,

$$\Theta^* f(x,t) = f(\Theta(\xi,\tau)), \quad \Theta_* g(\xi,\tau) = g(\Theta^{-1}(x,t)). \tag{1.4}$$

Remark 1.1. (1) Let f and g be functions defined on $\mathbf{R}^2$, and

$$\widehat{f}(y') = \int_{\mathbf{R}^2} e^{-i\xi'\cdot y'} f(\xi')\, d\xi', \quad \mathcal{F}^{-1}_{y'}[g](\xi') = \frac{1}{(2\pi)^2} \int_{\mathbf{R}^2} e^{i\xi'\cdot y'} g(y')\, dy'.$$

Then the solution of (1.3) is given by

$$H(\xi,\tau) = \mathcal{E}[h(\cdot,\tau)](\xi), \quad \mathcal{E}[f](\xi) = \mathcal{F}^{-1}_{y'}[e^{|y'|\xi_3} \widehat{f}(y')](\xi') \ (\xi_3 < 0). \tag{1.5}$$

(2) Θ define a C^1-diffeomorphism if h and H have the regularity described in Theorem 2.1.

Set $\mathbf{u} = \mathbf{u}(\xi,\tau) = \Theta^*\mathbf{v}(x,t)$ and $\theta = \theta(\xi,\tau) = \Theta^*\pi(x,t)$, and apply Θ^* to the 1st, 2nd, 3rd, and 4th line from the left-hand side. The system (1.2) are then reduced to

$$\begin{cases} \partial_\tau \mathbf{u} - \Delta\mathbf{u} + \nabla\theta = \mathbf{F}(\mathbf{u},H) & \text{in } \mathbf{R}^3_-,\ \tau > 0, \\ \operatorname{div}\mathbf{u} = F_d(\mathbf{u},H) = \operatorname{div}\mathbf{F}_d(\mathbf{u},H) & \text{in } \mathbf{R}^3_-,\ \tau > 0, \\ \mathbf{S}(\mathbf{u},\theta)\mathbf{e}_3 + (c_g - c_\sigma\Delta')h\mathbf{e}_3 = \mathbf{G}(\mathbf{u},H) & \text{on } \mathbf{R}^3_0,\ \tau > 0, \\ \partial_\tau h - \mathbf{u}\cdot\mathbf{e}_3 = G_h(\mathbf{u},H) & \text{on } \mathbf{R}^3_0,\ \tau > 0, \\ \mathbf{u}|_{t=0} = \mathbf{u}_0 & \text{in } \mathbf{R}^3_-, \\ h|_{t=0} = h_0 & \text{on } \mathbf{R}^2, \end{cases} \tag{1.6}$$

where we have set $\rho = \mu = 1$ without loss of generality.

Here $\mathbf{u}_0 = \mathbf{u}_0(\xi) = \Theta_0^*\mathbf{v}_0(x) = \mathbf{v}_0(\Theta_0(\xi))$ with $\Theta_0(\xi) = (\xi_1, \xi_2, \xi_3 + H_0(\xi))$ for

$$H_0(\xi) = \mathcal{E}[h_0](\xi), \tag{1.7}$$

where $\mathcal{E}$ are defined as (1.5), and the right members $\mathbf{F}$, F_d, $\mathbf{F}_d$, $\mathbf{G}$, and G_h are nonlinear terms with respect to $\mathbf{u}$ and H (cf. [14, Section 4.2] for the detail).

The goal of this article is to show the global well-posedness of (1.6) in the L_p in time and L_q in space setting. Here and subsequently, such a setting is called the L_p-L_q framework, and the main result is introduced in the next section. We suppose that exponents p, q satisfy the condition:

$$2 < p < \infty, \quad 3 < q < \frac{16}{5}, \quad \frac{2}{p} + \frac{3}{q} < 1, \tag{1.8}$$

which plays an important role when we solve (1.6) by using the contraction mapping theorem in Section 5. Note that we can not take $p = q$ satisfying (1.8). In fact, if we set $p = q$ in (1.8), then

$$\frac{2}{p} + \frac{3}{q} < 1 \Rightarrow 5 < q \Rightarrow \text{there is no intersection with } 3 < q < \frac{16}{5}.$$

This tells us that the L_p in time and L_q in space setting is essential in our approach.

1.3. **Historical remarks.** Beale considered the incompressible Navier-Stokes equations in $\Omega(t) = \{(x', x_3) \mid x' \in \mathbf{R}^2, \ -b < x_3 < h(x', t)\}$ for some $b > 0$ in [5]. More precisely, he showed the local well-posedness in the L_2-L_2 framework under the condition: $c_\sigma = 0$ and $c_g > 0$. Concerning the same $\Omega(t)$ as mentioned above, there are many results as follows: The global well-posedness was proved in Beale [6] under the smallness condition for initial data by taking into account $c_\sigma > 0$, and Beale and Nishida [7] showed polynomial decay of the solution obtained in [6]. Although [7] is a survey article, we can see the detailed proof in Hataya [9]. Along with these studies, we also refer e.g. to Allain [3], Tani and Tanaka [20], Tani [19], Hataya and Kawashima [10], and Bae [4]. We note that all of these results were proved in the L_2-L_2 framework. In the L_p-L_q framework, there are results of the local well-posendness due to Abels [1] with $p = q$ and Shibata [17], whereas Saito [15] showed the maximal L_p-L_q regularity theorem of some linearized system.

In the case of (1.1) with $c_g > 0$, Prüss and Simonett showed the local well-posedness in the L_p-L_p framework for both $c_\sigma > 0$ and $c_\sigma = 0$ in [11], [12], and [13]. They originally considered two-phase free boundary problems of the incompressible Navier-Stokes equations, but (1.1) was contained in their situations. In the L_p-L_q framework, Shibata and Shimizu [18] showed the maximal L_p-L_q regularity theorem for the linearized problem of (1.6).

On the other hand, we show in this article the global well-posedness of (1.6) in the L_p-L_q framework, and also we want to emphasize that the L_p-L_q framework is essential in our approach, as was seen in the condition (1.8).

This article is organized as follows: The next section tells us notation and main results of this article. Section 3 shows decay properties of solutions to the Stokes equations with surface tension and gravity. In Section 4, we state some proposition, concerning the full linearized system of (1.6), which is proved by the maximal L_p-L_q regularity property and the decay properties introduced in Section 3. Section 5 proves our main result, i.e. the global well-poseness of (1.6) in the L_p-L_q framework.

2. Notation and Main results

In this section, we first introduce notation used throughout this article. Next our main result is stated.

2.1. **Notation.** Let X be a Banach space and $\|\cdot\|_X$ its norm. In addition, let $\Omega \subset \mathbf{R}^n$ ($n \in \mathbf{N}$) be a domain. The following notation is used throughout this article:

- For $1 \le p \le \infty$ and $m \in \mathbf{N}$, $L_p(\Omega, X)$ and $W_p^m(\Omega, X)$ denote the X-valued Lebesgue and Sobolev spaces on Ω, respectively, and $L_p(\Omega) := L_p(\Omega, \mathbf{R})$ and $W_p^m(\Omega) := W_p^m(\Omega, \mathbf{R})$.
- $W_p^0(\Omega, X) := L_p(\Omega, X)$ and $W_p^0(\Omega) := L_p(\Omega)$.
- Let $1 \le p < \infty$ and $s \in (0, \infty) \setminus \mathbf{N}$. Then $W_p^s(\Omega, X)$ is the X-valued Sobolev-Slobodeckii space on Ω, that is, for $[s] = \max\{l \in \mathbf{N} \cup \{0\} \mid l < s\}$,

$$W_p^s(\Omega, X) = \Bigg\{ f \in W_p^{[s]}(\Omega, X) \mid \|f\|_{W_p^s(\Omega,X)} = \|f\|_{W_p^{[s]}(\Omega,X)} + \sum_{|\alpha|=[s]} \left(\int_\Omega \int_\Omega \frac{\|D^\alpha f(x) - D^\alpha f(y)\|_X^p}{|x-y|^{n+(s-[s])p}} \, dxdy \right)^{1/p} < \infty \Bigg\},$$

  where $D^\alpha f(x) = \partial_1^{\alpha_1} \dots \partial_n^{\alpha_n} f(x)$ for multi-index $\alpha = (\alpha_1, \dots, \alpha_n) \in (\mathbf{N} \cup \{0\})^n$. In addition, $W_p^s(\Omega) := W_p^s(\Omega, \mathbf{R})$.
- Let $1 \le p, q < \infty$ and $0 \le s_0, s_1 < \infty$ with $s_0 \ne s_1$. Then we set

$$B_{q,p}^s(\Omega) = (W_q^{s_0}, W_q^{s_1})_{\theta,p} \quad (0 < \theta < 1, \ \ s = (1-\theta)s_0 + \theta s_1),$$

where $(\cdot,\cdot)_{\theta,p}$ is the real interpolation functor (cf. [21, Theorem 3.3.6], [8, Theorem 6.2.4]).

- Let $1 < p < \infty$, and we set $\widehat{W}^1_p(\Omega) = \{\theta \in L_{1,\mathrm{loc}}(\Omega) \mid \nabla\theta \in L_p(\Omega)^n\}$.
- Let Y be another Banach space. Then $\mathcal{B}(X,Y)$ denotes the Banach space of all bounded linear operators from X to Y.
- For any 3-vector $\mathbf{f}$ defined on $\mathbf{R}^3_0$, we set $[\mathbf{f}]_{\tan} = \mathbf{f} - (\mathbf{f}\cdot\mathbf{e}_3)\mathbf{e}_3$.
- Let $m \in \mathbf{N}$ and $\mathbf{a}\cdot\mathbf{b} = \sum_{j=1}^m a_j b_j$ for m-vectors $\mathbf{a} = (a_1,\cdots,a_m)^T$ and $\mathbf{b} = (b_1,\dots,b_m)^T$. In addition, we set $(\mathbf{f},\mathbf{g})_\Omega = \int_\Omega \mathbf{f}(x)\cdot\mathbf{g}(x)\,dx$ for m-vector functions $\mathbf{f},\mathbf{g}$ on Ω.
- The letter C denotes a generic constant and $C(a,b,c,\dots)$ a generic constant depending on the quantities $a,b,c,\dots$. The value of C and $C(a,b,c,\dots)$ may change from line to line.

2.2. **Main results.** Let $\mathbb{I}_1$ and $\mathbb{I}_2$ be

$$
\begin{aligned}
\mathbb{I}_1 &= \left(B^{2(1-1/p)}_{q,p}(\mathbf{R}^3_-) \cap B^{2(1-1/p)}_{q/2,p}(\mathbf{R}^3_-)\right)^3, \\
\mathbb{I}_2 &= B^{3-1/p-1/q}_{q,p}(\mathbf{R}^2) \cap B^{3-1/p-1/2}_{2,p}(\mathbf{R}^2) \cap L_{q/2}(\mathbf{R}^2),
\end{aligned} \tag{2.1}
$$

which are functions spaces for the initial data $\mathbf{u}_0$ and h_0, respectively. Our main result is then stated as follows:

Theorem 2.1. *Let exponents p,q satisfy (1.8), $c_g > 0$, and $c_\sigma > 0$. Suppose that $(\mathbf{u}_0,h_0) \in \mathbb{I}_1 \times \mathbb{I}_2$ and H_0 is given by (1.7). Then there exist positive constants ε_0 and δ_0 sufficiently small, depending only on p,q,c_g, and c_σ, such that the equations (1.6) and (1.3) admits a unique solution $(\mathbf{u},\theta,h,H)$ in X_{δ_0}, where X_{δ_0} is defined as in Section 5, if the initial data $(\mathbf{u}_0,h_0)$ satisfies the smallness condition: $\|(\mathbf{u}_0,h_0)\|_{\mathbb{I}_1\times\mathbb{I}_2} < \varepsilon_0$ and the compatibility conditions:*

$$
\operatorname{div}\mathbf{u}_0 = F_d(\mathbf{u}_0,H_0) \quad \text{in } \mathbf{R}^3_-, \qquad [\mathbf{D}(\mathbf{u}_0)\mathbf{e}_3]_{\tan} = [\mathbf{G}(\mathbf{u}_0,H_0)]_{\tan} \quad \text{on } \mathbf{R}^3_0.
$$

Remark 2.2. If we set $\mathbf{v} = \Theta_*\mathbf{u}$ and $\pi = \Theta_*\theta$ by (1.4), where $\mathbf{u}$ and θ is the solution obtained in Theorem 2.1, then $(\mathbf{v},\pi,h)$ solves (1.1).

3. Decay properties of solutions to the Stokes equations

In this section, we are concerned with decay properties of solutions to the following Stokes equations with surface tension and gravity:

$$
\begin{cases}
\partial_t\mathbf{u} - \Delta\mathbf{u} + \nabla\theta = 0 & \text{in } \mathbf{R}^3_-,\ t>0, \\
\operatorname{div}\mathbf{u} = 0 & \text{in } \mathbf{R}^3_-,\ t>0, \\
\mathbf{S}(\mathbf{u},\theta)\mathbf{e}_3 + (c_g - c_\sigma\Delta')h\mathbf{e}_3 = 0 & \text{on } \mathbf{R}^3_0,\ t>0, \\
\partial_t h - \mathbf{u}\cdot\mathbf{e}_3 = 0 & \text{on } \mathbf{R}^3_0,\ t>0, \\
\mathbf{u}|_{t=0} = \mathbf{f} & \text{in } \mathbf{R}^3_-,\ t>0, \\
h|_{t=0} = g & \text{on } \mathbf{R}^3_0.
\end{cases} \tag{3.1}
$$

To show the decay properties, we introduce some function spaces here. Let $1 < q < \infty$ and $\widehat{W}^1_{q,0}(\mathbf{R}^3_-) = \{\theta \in \widehat{W}^1_q(\mathbf{R}^3_-) \mid \theta|_{\mathbf{R}^3_0} = 0\}$, and

$$
J_q(\mathbf{R}^3_-) = \{\mathbf{u} \in L_q(\mathbf{R}^3_-)^3 \mid (\mathbf{u},\nabla\varphi)_{\mathbf{R}^3_-} = 0 \text{ for any } \varphi \in \widehat{W}^1_{q',0}(\mathbf{R}^3_-)\},
$$

where $1/q + 1/q' = 1$. For simplicity, we set

$$
X_q = J_q(\mathbf{R}^3_-) \times W^{2-1/q}_q(\mathbf{R}^2), \quad X^0_q = L_q(\mathbf{R}^3_-) \times L_q(\mathbf{R}^2), \quad X^2_q = L_q(\mathbf{R}^3_-) \times W^{2-1/q}_q(\mathbf{R}^2),
$$

and, for $1 \le s \le 2 \le r \le \infty$,

$$\begin{aligned}(3.2)\qquad m(s,r) &= \left(\frac{1}{s}-\frac{1}{r}\right)+\frac{1}{2}\left(\frac{1}{2}-\frac{1}{r}\right),\\ n(s,r) &= \left(\frac{1}{s}-\frac{1}{r}\right)+\min\left\{\frac{1}{2}\left(\frac{1}{s}-\frac{1}{r}\right),\frac{1}{8}\left(2-\frac{1}{r}\right)\right\}.\end{aligned}$$

Then the following proposition holds (cf. [16, Theorem 1.1] and [14, Theorem 3.1.1 and Theorem 3.1.3]).

Proposition 3.1. *Let $1 < q < \infty, c_g > 0$, and $c_\sigma > 0$.*

(1) *For every $t > 0$, there exists operators*

$$S(t) \in \mathcal{B}(X_q^2, W_q^2(\mathbf{R}_-^3)^3), \quad \Pi(t) \in \mathcal{B}(X_q^2, \widehat{W}_q^1(\mathbf{R}_-^3)), \quad T(t) \in \mathcal{B}(X_q^2, W_q^{3-1/q}(\mathbf{R}^2))$$

such that, for any $\mathbf{F} = (\mathbf{f}, g) \in X_q$,

$$\begin{aligned}&S(\cdot)\mathbf{F} \in C^1((0,\infty), J_q(\mathbf{R}_-^3)) \cap C((0,\infty), W_q^2(\mathbf{R}_-^3)^3),\\ &\Pi(\cdot)\mathbf{F} \in C((0,\infty), \widehat{W}_p^1(\mathbf{R}_-^3)),\\ &T(\cdot)\mathbf{F} \in C^1((0,\infty), W_q^{2-1/q}(\mathbf{R}^2)) \cap C((0,\infty), W_q^{3-1/q}(\mathbf{R}^2)),\end{aligned}$$

and $(\mathbf{u}, \theta, h) = (S(t)\mathbf{F}, \Pi(t)\mathbf{F}, T(t)\mathbf{F})$ solves uniquely the system (3.1) with

$$\lim_{t\to 0+} \|(\mathbf{u}(t), h(t)) - (\mathbf{f}, g)\|_{X_q} = 0.$$

(2) *Let $1 \le s \le 2 \le r \le \infty$ and $\mathbf{F} = (\mathbf{f}, g) \in X_s^0 \cap X_q^2$. The operators obtained in (1) are decomposed into*

$$\begin{aligned}S(t)\mathbf{F} &= S_0(t)\mathbf{F} + S_\infty(t)\mathbf{F} + R(t)\mathbf{f},\\ \Pi(t)\mathbf{F} &= \Pi_0(t)\mathbf{F} + \Pi_\infty(t)\mathbf{F} + P(t)\mathbf{f},\\ T(t)\mathbf{F} &= T_0(t)\mathbf{F} + T_\infty(t)\mathbf{F},\end{aligned}$$

which satisfy the following estimates: First, for $k = 1, 2$, $l = 0, 1, 2$, and $t \ge 1$,

$$\begin{aligned}\|(S_0(t)\mathbf{F}, \partial_t\mathcal{E}(T_0(t)\mathbf{F}))\|_{L_r(\mathbf{R}_-^3)} &\le C(t+2)^{-m(s,r)}\|\mathbf{F}\|_{X_s^0} \quad \text{if } (r,s) \ne (2,2),\\ \|\nabla^k S_0(t)\mathbf{F}\|_{L_r(\mathbf{R}_-^3)} &\le C(t+2)^{-n(s,r)-\frac{k}{8}}\|\mathbf{F}\|_{X_s^0},\\ \|(\partial_t S_0(t)\mathbf{F}, \nabla\Pi_0(t)\mathbf{F})\|_{L_r(\mathbf{R}_-^3)} &\le C(t+2)^{-m(s,r)-\frac{1}{4}}\|\mathbf{F}\|_{X_s^0},\\ \|\nabla^k\partial_t\mathcal{E}(T_0(t)\mathbf{F})\|_{L_r(\mathbf{R}_-^3)} &\le C(t+2)^{-m(s,r)-\frac{k}{2}}\|\mathbf{F}\|_{X_s^0},\\ \|\nabla^{1+l}\mathcal{E}(T_0(t)\mathbf{F})\|_{L_r(\mathbf{R}_-^3)} &\le C(t+2)^{-m(s,r)-\frac{1}{4}-\frac{l}{2}}\|\mathbf{F}\|_{X_s^0},\\ \|T_0(t)\mathbf{F}\|_{L_r(\mathbf{R}^2)} &\le C(t+2)^{-(\frac{1}{s}-\frac{1}{r})}\|\mathbf{F}\|_{X_s^0} \quad \text{if } s \ne 2\end{aligned}$$

with some positive constant C, where $\mathcal{E}$ is defined as (1.5). Secondly, there exist positive constants γ and C such that, for every $t \ge 1$,

$$\begin{aligned}&\|\partial_t S_\infty(t)\mathbf{F}\|_{L_q(\mathbf{R}_-^3)} + \|S_\infty(t)\mathbf{F}\|_{W_q^2(\mathbf{R}_-^3)} + \|\Pi_\infty(t)\mathbf{F}\|_{W_q^1(\mathbf{R}_-^3)}\\ &\quad + \|\partial_t\mathcal{E}(T_\infty(t)\mathbf{F}))\|_{W_q^2(\mathbf{R}_-^3)} + \|\mathcal{E}(T_\infty(t)\mathbf{F})\|_{W_q^3(\mathbf{R}_-^3)} \le Ce^{-\gamma t}\|\mathbf{F}\|_{X_q^2}.\end{aligned}$$

Thirdly, there is a positive constant C such that, for every $t \geq 1$ and $l = 0, 1, 2$,

$$\|\nabla^l R(t)\mathbf{f}\|_{L_q(\mathbf{R}^3_-)} \leq C(t+2)^{-\frac{l}{2}}\|\mathbf{f}\|_{L_q(\mathbf{R}^3_-)},$$
$$\|(\partial_t R(t)\mathbf{f}, \nabla P(t)\mathbf{f})\|_{L_q(\mathbf{R}^3_-)} \leq C(t+2)^{-1}\|\mathbf{f}\|_{L_q(\mathbf{R}^3_-)}.$$

4. Full linearized problem

We consider in this section the full linearized system of (1.6) as follows:

$$\begin{cases} \partial_t \mathbf{u} - \Delta\mathbf{u} + \nabla\theta = \mathbf{f}_1 + \mathbf{f}_2 & \text{in } \mathbf{R}^3_-,\ t > 0, \\ \operatorname{div}\mathbf{u} = f_d = \operatorname{div}\mathbf{f}_d & \text{in } \mathbf{R}^3_-,\ t > 0, \\ \mathbf{S}(\mathbf{u}, \theta)\mathbf{e}_3 + (c_g - c_\sigma\Delta')h\mathbf{e}_3 = \mathbf{g} & \text{on } \mathbf{R}^3_0,\ t > 0, \\ \partial_t h - \mathbf{u}\cdot\mathbf{e}_3 = g_h & \text{on } \mathbf{R}^3_0,\ t > 0, \\ \mathbf{u}|_{t=0} = \mathbf{u}_0 & \text{in } \mathbf{R}^3_-, \\ h|_{t=0} = h_0 & \text{on } \mathbf{R}^2. \end{cases} \tag{4.1}$$

We here introduce some symbols to state main results of this section precisely.

First, let $s > 0$ and $1 \leq p \leq \infty$, and let X be a Banach space and its norm $\|\cdot\|_X$. We then set

$$L^s_p((0,\infty), X) = \{f \in L_p((0,\infty), X) \mid \|f\|_{L^s_p((0,\infty),X)} < \infty\},$$
$$\|f\|_{L^s_p((0,\infty),X)} = \|(t+2)^s f\|_{L_p((0,\infty),X)},$$
$$W^{1,s}_p((0,\infty), X) = \{f \in W^1_p((0,\infty), X) \mid \|f\|_{W^{1,s}_p((0,\infty),X)} < \infty\},$$
$$\|f\|_{W^{1,s}_p((0,\infty),X)} = \|\partial_t((t+2)^s f)\|_{L_p((0,\infty),X)}.$$

Secondly, we define a function space $\widehat{W}^{-1}_q(\mathbf{R}^3_-)$. Let E be the extension operator given by [2, Theorem 5.19], and we set

$$\widehat{W}^{-1}_q(\mathbf{R}^3_-) = \{f \in L_{1,\mathrm{loc}}(\mathbf{R}^3_-) \mid (1-\Delta)^{-1/2}Ef \in L_q(\mathbf{R}^3_-)\},$$
$$\|f\|_{\widehat{W}^{-1}_q(\mathbf{R}^3_-)} = \|(1-\Delta)^{-1/2}Ef\|_{L_q(\mathbf{R}^3)},$$

where $(1-\Delta)^{-1/2}u = \mathcal{F}^{-1}_\xi[(1+|\xi|^2)^{-1/2}\widehat{u}(\xi)](x)$ for functions $u = u(x)$ on $\mathbf{R}^3$.

Thirdly, we introduce function spaces for the right members of (4.1). Let exponents p, q satisfy (1.8). We then set

$$\mathbb{F}_1 = \mathbb{F}_2 = \bigcap_{r\in\{q,2\}} L_p((0,\infty), L_r(\mathbf{R}^3_-))^3, \quad \mathbb{G}_h = \bigcap_{r\in\{q,2\}} W^{2,1}_{r,p}(\mathbf{R}^3_- \times (0,\infty)),$$
$$\mathbb{F}_{d1} = \bigcap_{r\in\{q,2\}} W^1_p((0,\infty), L_r(\mathbf{R}^3_-))^3, \qquad \mathbb{F}_{d2} = \bigcap_{r\in\{q,2\}} L_p((0,\infty), W^1_r(\mathbf{R}^3_-)),$$
$$\mathbb{G} = \bigcap_{r\in\{q,2\}} W^1_p((0,\infty), \widehat{W}^{-1}_r(\mathbf{R}^3_-))^3 \cap L_p((0,\infty), W^1_r(\mathbf{R}^3_-))^3,$$

where $W^{2,1}_{r,p}(\mathbf{R}^3_- \times (0,\infty)) = W^1_p((0,\infty), L_r(\mathbf{R}^3_-)) \cap L_p((0,\infty), W^2_r(\mathbf{R}^3_-))$, and furthermore, for $\delta > 0$ and $\varepsilon > 0$

$$\widetilde{\mathbb{F}}_1(\delta,\varepsilon) = L^\delta_p((0,\infty), L_q(\mathbf{R}^3_-))^3 \cap L^\varepsilon_\infty((0,\infty), L_{q/2}(\mathbf{R}^3_-))^3,$$
$$\widetilde{\mathbb{F}}_2(\delta,\varepsilon) = L^\delta_p((0,\infty), L_q(\mathbf{R}^3_-))^3 \cap L^\varepsilon_p((0,\infty), L_{q/2}(\mathbf{R}^3_-))^3,$$
$$\widetilde{\mathbb{G}}_h(\delta,\varepsilon) = L^\delta_p((0,\infty), W^2_q(\mathbf{R}^3_-)) \cap L^\varepsilon_p((0,\infty), W^2_{q/2}(\mathbf{R}^3_-)),$$

$$
\begin{aligned}
&\widetilde{\mathbb{F}}_{d1}(\delta,\varepsilon) = W^{1,\delta}_p((0,\infty), L_q(\mathbf{R}^3_-))^3 \cap W^{1,\varepsilon}_p((0,\infty), L_{q/2}(\mathbf{R}^3_-))^3, \\
&\widetilde{\mathbb{F}}_{d2}(\delta,\varepsilon) = L^{\delta}_p((0,\infty), W^1_q(\mathbf{R}^3_-)) \cap L^{\varepsilon}_p((0,\infty), W^1_{q/2}(\mathbf{R}^3_-)), \\
&\widetilde{\mathbb{G}}(\delta,\varepsilon) = W^{1,\delta}_p((0,\infty), \widehat{W}^{-1}_q(\mathbf{R}^3_-))^3 \cap L^{\delta}_p((0,\infty), W^1_q(\mathbf{R}^3_-))^3 \\
&\qquad \cap W^{1,\varepsilon}_p((0,\infty), \widehat{W}^{-1}_{q/2}(\mathbf{R}^3_-))^3 \cap L^{\varepsilon}_p((0,\infty), W^1_{q/2}(\mathbf{R}^3_-))^3.
\end{aligned}
$$

Moreover, we define additional function spaces as

$$
\begin{aligned}
\mathbb{A}_1 &= L^{m(q/2,q)}_\infty((0,\infty), L_q(\mathbf{R}^3_-)) \cap L^{m(q/2,2)}_\infty((0,\infty), L_2(\mathbf{R}^3_-)), \\
\mathbb{A}_2 &= L^{m(q/2,q)+1/2}_\infty((0,\infty), L_q(\mathbf{R}^3_-)) \cap L^{m(q/2,2)+1/2}_\infty((0,\infty), L_2(\mathbf{R}^3_-)), \\
\widehat{\mathbb{A}}_2 &= L^{m(q/2,q)+1/2}_\infty((0,\infty), \widehat{W}^1_q(\mathbf{R}^3_-)) \cap L^{m(q/2,2)+1/2}_\infty((0,\infty), \widehat{W}^1_2(\mathbf{R}^3_-)), \\
\mathbb{A}_3 &= L^{m(q/2,q)+1/2}_p((0,\infty), W^1_q(\mathbf{R}^3_-)) \cap L^{m(q/2,2)+1/2}_p((0,\infty), W^1_2(\mathbf{R}^3_-)).
\end{aligned}
$$

Finally, we introduce the following three norms: Let p, q satisfy (1.8) and $r \in \{q, 2\}$, and we set

$$
\begin{aligned}
\mathbb{D}_r(\mathbf{u}, h, \partial_t h, H) &= \|\mathbf{u}\|_{L^{m(q/2,r)}_\infty((0,\infty),L_r(\mathbf{R}^3_-))} + \|\nabla\mathbf{u}\|_{L^{n(q/2,r)+1/8}_\infty((0,\infty),L_r(\mathbf{R}^3_-))} \\
&+ \|h\|_{L^{2/q-1/r}_\infty((0,\infty),L_r(\mathbf{R}^2))} + \|\partial_t h\|_{L^{m(q/2,r)}_\infty((0,\infty),L_r(\mathbf{R}^2))} \\
&+ \|\nabla H\|_{L^{m(q/2,r)+1/4}_\infty((0,\infty),W^1_r(\mathbf{R}^3_-))} + \|\nabla\partial_t H\|_{L^{m(q/2,r)+1/2}_\infty((0,\infty),L_r(\mathbf{R}^3_-))},
\end{aligned}
$$

which is used to control decay properties of the lower order terms. In addition,

$$
\begin{aligned}
\mathbb{M}_{r,p}(\mathbf{u}, \theta, h, \partial_t h, H) &= \|(\partial_t\mathbf{u}, \mathbf{u}, \nabla\mathbf{u}, \nabla^2\mathbf{u}, \nabla\theta)\|_{L_p((0,\infty),L_r(\mathbf{R}^3_-))} \\
&+ \|h\|_{L_p((0,\infty),W^{3-1/r}_r(\mathbf{R}^2))} + \|\partial_t h\|_{L_p((0,\infty),W^{2-1/r}_r(\mathbf{R}^2))} \\
&+ \|\nabla H\|_{L_p((0,\infty),W^2_r(\mathbf{R}^3_-))} + \|\nabla\partial_t H\|_{L_p((0,\infty),W^1_r(\mathbf{R}^3_-))},
\end{aligned}
$$

where $\mathbb{M}$ stands for *maximal regularity*. For the highest order terms, we additionally set a weighted norm:

$$
\mathbb{W}_{q,p}(\mathbf{u}, H; \delta_1, \delta_2) = \|(\partial_t\mathbf{u}, \nabla^2\mathbf{u})\|_{L^{\delta_1}_p((0,\infty),L_q(\mathbf{R}^3_-))} + \|(\nabla^2\partial_t H, \nabla^3 H)\|_{L^{\delta_2}_p((0,\infty),L_q(\mathbf{R}^3_-))}.
$$

The main result of this section is then stated as follows:

Proposition 4.1. *Let exponents p, q satisfy* (1.8), *and $c_g > 0$ and $c_\sigma > 0$. Let $\varepsilon_1 > 1, \varepsilon_2 \geq 1$, and $\varepsilon_3 \geq 1$, and also $0 < \delta_1, \delta_2 \leq 1$ satisfy the conditions:*

$$
\begin{aligned}
&p\left(\min\{\varepsilon_1,\varepsilon_2,\varepsilon_3\} - \delta_1\right) > 1, \quad p\left(m\left(\frac{q}{2}, q\right) + \frac{1}{4} - \delta_1\right) > 1, \\
&p\left(\min\{\varepsilon_1,\varepsilon_2,\varepsilon_3\} - \delta_2\right) > 1, \quad p\left(m\left(\frac{q}{2}, 2\right) + 1 - \delta_2\right) > 1.
\end{aligned}
\tag{4.2}
$$

We set $\delta_0 = \max\{\delta_1, \delta_2\}$ and suppose that the right members of the system (4.1) *satisfy the following conditions:*

(1) $\mathbf{f}_1 \in \mathbb{F}_1 \cap \widetilde{\mathbb{F}}_1(\delta_0, \varepsilon_1)$;
(2) $\mathbf{f}_2 \in \mathbb{F}_2 \cap \widetilde{\mathbb{F}}_2(\delta_0, \varepsilon_2)$;
(3) $g_h \in \mathbb{G}_h \cap \widetilde{\mathbb{G}}_h(\delta_0, \varepsilon_3) \cap \mathbb{A}_1 \cap \widehat{\mathbb{A}}_2$;
(4) $\mathbf{f}_d \in \mathbb{F}_{d1} \cap \widetilde{\mathbb{F}}_{d1}(\delta_0, \varepsilon_2) \cap \widetilde{\mathbb{F}}_{d1}(\delta_0, \varepsilon_3) \cap \mathbb{A}_1$;
(5) $f_d \in \mathbb{F}_{d2} \cap \widetilde{\mathbb{F}}_{d2}(\delta_0, \varepsilon_2) \cap \widetilde{\mathbb{F}}_{d2}(\delta_0, \varepsilon_3) \cap \mathbb{A}_2 \cap \mathbb{A}_3$;
(6) $\mathbf{g} \in \mathbb{G} \cap \widetilde{\mathbb{G}}(\delta_0, \varepsilon_2) \cap \widetilde{\mathbb{G}}(\delta_0, \varepsilon_3) \cap \mathbb{A}_3$;

(7) f_d *and* $\mathbf{g}$ *satisfy additionally*

$$(f_d, \mathbf{g}) \in \left(L_p^{\alpha}((0,\infty), W_q^1(\mathbf{R}_-^3)) \cap L_p^{\beta}((0,\infty), W_{q/2}^1(\mathbf{R}_-^3))\right)^4$$

with some positive numbers α *and* β *satisfying*

$$p(1+\alpha-\delta_0) > 1, \quad p(1+\beta-\max\{\varepsilon_2, \varepsilon_3\}) > 1; \tag{4.3}$$

(8) $(\mathbf{u}_0, h_0) \in \mathbb{I}_1 \times \mathbb{I}_2$ *satisfies the compatibility conditions:*

$$f_d|_{t=0} = \operatorname{div} \mathbf{u}_0 \quad \text{in } \mathbf{R}_-^3, \quad [\mathbf{g}]_{\tan} = [\mathbf{D}(\mathbf{u}_0)\mathbf{e}_3]_{\tan} \quad \text{on } \mathbf{R}_0^3.$$

Then there exists a unique solution $(\mathbf{u}, \theta, h, H)$ *of the equations* (4.1) *and* (1.3), *which satisfies*

$$\begin{aligned}
&\sum_{r\in\{q,2\}} \Big(\mathbb{D}_r(\mathbf{u}, h, \partial_t h, H) + \mathbb{M}_{r,p}(\mathbf{u}, \theta, h, \partial_t h, H)\Big) + \mathbb{W}_{q,p}(\mathbf{u}, H; \delta_1, \delta_2) \\
&\le C(p,q)\Big(\|(\mathbf{u}_0, h_0)\|_{\mathbb{I}_1\times\mathbb{I}_2} + \|\mathbf{f}_1\|_{\mathbb{F}_1\cap\widetilde{\mathbb{F}}_1(\delta_0,\varepsilon_1)} + \|\mathbf{f}_2\|_{\mathbb{F}_2\cap\widetilde{\mathbb{F}}_2(\delta_0,\varepsilon_2)} \\
&+ \|g_h\|_{\mathbb{G}_h\cap\widetilde{\mathbb{G}}_h(\delta_0,\varepsilon_3)\cap\mathbb{A}_1\cap\widehat{\mathbb{A}}_2} + \|\mathbf{f}_d\|_{\mathbb{F}_{d1}\cap\widetilde{\mathbb{F}}_{d1}(\delta_0,\varepsilon_2)\cap\widetilde{\mathbb{F}}_{d1}(\delta_0,\varepsilon_3)\cap\mathbb{A}_1} \\
&+ \|f_d\|_{\mathbb{F}_{d2}\cap\widetilde{\mathbb{F}}_{d2}(\delta_0,\varepsilon_2)\cap\widetilde{\mathbb{F}}_{d2}(\delta_0,\varepsilon_3)\cap\mathbb{A}_2\cap\mathbb{A}_3} + \|\mathbf{g}\|_{\mathbb{G}\cap\widetilde{\mathbb{G}}(\delta_0,\varepsilon_2)\cap\widetilde{\mathbb{G}}(\delta_0,\varepsilon_3)\cap\mathbb{A}_3} \\
&+ \|(f_d, \mathbf{g})\|_{L_p^{\alpha}((0,\infty),W_q^1(\mathbf{R}_-^3))\cap L_p^{\beta}((0,\infty),W_{q/2}^1(\mathbf{R}_-^3))}\Big)
\end{aligned}$$

with some positive constant $C(p,q)$.

Proof. We can prove the proposition by using Proposition 3.1. See [14, Theorem 4.4.1] for the detail. □

5. Proof of Theorem 2.1

Our aim in this section is to show Theorem 2.1. To use the contraction mapping theorem, we set, for $R > 0$,

$$\begin{aligned}
X_R = \Big\{\mathbf{z} = (\mathbf{u}, \theta, h, H) \mid \|\mathbf{z}\|_X := \sum_{r\in\{q,2\}} \Big(&\mathbb{D}_r(\mathbf{u}, h, \partial_t h, H) \\
&+ \mathbb{M}_{r,p}(\mathbf{u}, \theta, h, \partial_t h, H)\Big) + \mathbb{W}_{q,p}(\mathbf{u}, H; 1/2, 3/4) < R\Big\},
\end{aligned}$$

where $\mathbb{D}_r$, $\mathbb{M}_{r,p}$, and $\mathbb{W}_{q,p}$ are defined in Section 4.

We remind, in [14, Section 4.2], that the nonlinear term $\mathbf{F}(\mathbf{u}, H)$ is given by $\mathbf{F}(\mathbf{u}, H) = \mathbf{F}_1(\mathbf{u}, H) + \mathbf{F}_2(\mathbf{u}, H)$ with

$$\begin{aligned}
\mathbf{F}_1(\mathbf{u}, H) &= (\mathbf{I} + \mathbf{M}_3(H))\left(\frac{\partial_t H \partial_3 \mathbf{u}}{1+\partial_3 H} - (\mathbf{u}\cdot\nabla)\mathbf{u}\right), \\
\mathbf{F}_2(\mathbf{u}, H) &= (-\partial_t u_3 + \Delta u_3)\nabla H + (\mathbf{I} + \mathbf{M}_3(H))\left(\sum_{j=1}^3 \mathcal{F}_{jj}(H)\mathbf{u} + \frac{(\mathbf{u}\cdot\nabla H)\partial_3\mathbf{u}}{1+\partial_3 H}\right),
\end{aligned}$$

where $\mathbf{M}(H) = (M_{ij}(H))$ is a 3×3 matrix with $M_{i1}(H) = 0$, $M_{i2}(H) = 0$, and $M_{i3}(H) = D_i H$ $(i = 1, 2, 3)$;

$$\begin{aligned}\mathcal{F}_{jk}(H) = &\frac{1}{(1+\partial_3 H)^3}\{(\partial_j\partial_k H)(1+\partial_3 H)^2 - (\partial_k H)(\partial_j\partial_3 H)(1+\partial_3 H)\\ &- (\partial_j H)(\partial_3\partial_k H)(1+\partial_3 H) + (\partial_j H)(\partial_k H)(\partial_3^2 H)\}\partial_3\\ &+ \left(\frac{\partial_k H}{1+\partial_3 H}\right)\partial_j\partial_3 + \left(\frac{\partial_j H}{1+\partial_3 H}\right)\partial_3\partial_k - \frac{(\partial_j H)(\partial_k H)}{(1+\partial_3 H)^2}\partial_3^2.\end{aligned}$$

Proof of Theorem 2.1 To show Theorem 2.1, we apply Proposition 4.1 with

$$\begin{aligned}&\varepsilon_1 = m\left(\frac{q}{2}, q\right) + n\left(\frac{q}{2}, q\right) + \frac{1}{8} = \frac{2}{q} + \frac{3}{8}, \quad \varepsilon_2 = \varepsilon_3 = 1,\\ &\delta_1 = \frac{1}{2}, \quad \delta_2 = \frac{3}{4}, \quad \alpha = 0, \quad \beta = \frac{1}{4},\end{aligned} \tag{5.1}$$

where m, n are defined as (3.2) and

$$n\left(\frac{q}{2}, q\right) = \frac{3}{2q}, \quad n\left(\frac{q}{2}, 2\right) = \frac{3}{2}\left(\frac{2}{q} - \frac{1}{2}\right) \tag{5.2}$$

under the assumption (1.8). We then note as follows: First the assumption $3 < q < 16/5$ implies that $\varepsilon_1 > 1$. Secondly, we see, by (1.8), that

$$p > 32, \quad p\left(1 + \alpha - \frac{3}{4}\right) = p(1 + \beta - 1) = \frac{p}{4} > 1, \tag{5.3}$$

which furnishes that the conditions (4.2) and (4.3) hold. Thirdly, (1.8) and Sobolev's embedding theorem yields that

$$\begin{aligned}&\|(\mathbf{u}, \nabla H)\|_{L_\infty((0,\infty), W^1_\infty(\mathbf{R}^3_-))} \leq M_1\|\mathbf{z}\|_X,\\ &\|(\mathbf{u}, \nabla H)\|_{L_\infty((0,\infty), W^1_q(\mathbf{R}^3_-))} \leq M_1\|\mathbf{z}\|_X,\\ &\|(\mathbf{u}, \nabla H)\|_{L_\infty((0,\infty), W^1_2(\mathbf{R}^3_-))} \leq M_1\|\mathbf{z}\|_X\end{aligned} \tag{5.4}$$

for $\mathbf{z} = (\mathbf{u}, \theta, h, H) \in X_{\delta_0}$ and some positive constant M_1 independent of $\mathbf{u}, H$, and $\mathbf{z}$. Here δ_0 is a positive number determined later.

Step 1 Our aim in this step is to show the following estimates:

$$\begin{aligned}&\|\mathbf{F}_1(\mathbf{u}, H)\|_{\mathbb{F}_1\cap\tilde{\mathbb{F}}_1(3/4, 2/q+3/8)} + \|\mathbf{F}_2(\mathbf{u}, H)\|_{\mathbb{F}_2\cap\tilde{\mathbb{F}}_2(3/4,1)} \leq C(p,q)\|\mathbf{z}\|_X^2,\\ &\|G_h(\mathbf{u}, H)\|_{\mathbb{G}_h\cap\tilde{\mathbb{G}}_h(3/4,1)\cap\mathbb{A}_1\cap\tilde{\mathbb{A}}_2} \leq C(p,q)\|\mathbf{z}\|_X^2,\\ &\|\mathbf{F}_d(\mathbf{u}, H)\|_{\mathbb{F}_{d1}\cap\tilde{\mathbb{F}}_{d1}(3/4,1)\cap\mathbb{A}_1} \leq C(p,q)\|\mathbf{z}\|_X^2,\\ &\|F_d(\mathbf{u}, H)\|_{\mathbb{F}_{d2}\cap\tilde{\mathbb{F}}_{d2}(3/4,1)\cap\mathbb{A}_2\cap\mathbb{A}_3} \leq C(p,q)\|\mathbf{z}\|_X^2,\\ &\|\mathbf{G}(\mathbf{u}, H)\|_{\mathbb{G}\cap\tilde{\mathbb{G}}(3/4,1)\cap\mathbb{A}_3} \leq C(p,q)\|\mathbf{z}\|_X^2,\\ &\|(F_d(\mathbf{u}, H), \mathbf{G}(\mathbf{u}, H))\|_{L_p((0,\infty), W^1_q(\mathbf{R}^3_-))\cap L_p^{1/4}((0,\infty), W^1_{q/2}(\mathbf{R}^3_-))} \leq C(p,q)\|\mathbf{z}\|_X^2\end{aligned} \tag{5.5}$$

for $\mathbf{z} = (\mathbf{u}, \theta, h, H) \in X_{\delta_0}$ with some positive constant $C(p, q)$. We only show the first line of (5.5) in the following. See [14, Theorem 4.5.1] for the other estimates.

We first consider $\mathbf{F}_1(\mathbf{u}, H)$. By (5.4) it is clear that, for $r \in \{q, 2\}$,

$$\|(\mathbf{u}\cdot\nabla)\mathbf{u}\|_{L_p((0,\infty), L_r(\mathbf{R}^3_-))} \leq \|\mathbf{u}\|_{L_\infty((0,\infty), L_\infty(\mathbf{R}^3_-))}\|\nabla\mathbf{u}\|_{L_p((0,\infty), L_r(\mathbf{R}^3_-))} \leq M_1\|\mathbf{z}\|_X^2, \tag{5.6}$$

and besides, Sobolev's embedding theorem and Hölder's inequality yield that

$$
\begin{aligned}
\|(\mathbf{u}(t)\cdot\nabla)\mathbf{u}(t)\|_{L_q(\mathbf{R}^3_-)} &\le \|\mathbf{u}(t)\|_{L_\infty(\mathbf{R}^3_-)}\|\nabla\mathbf{u}(t)\|_{L_q(\mathbf{R}^3_-)} \\
&\le C(q)\|\mathbf{u}(t)\|_{W^1_q(\mathbf{R}^3_-)}\|\nabla\mathbf{u}(t)\|_{L_q(\mathbf{R}^3_-)} \\
&\le C(q)(t+2)^{-(2/q+3/8)}\|\mathbf{z}\|_X^2, \\
\|(\mathbf{u}(t)\cdot\nabla)\mathbf{u}(t)\|_{L_{q/2}(\mathbf{R}^3_-)} &\le \|\mathbf{u}(t)\|_{L_q(\mathbf{R}^3_-)}\|\nabla\mathbf{u}(t)\|_{L_q(\mathbf{R}^3_-)} \\
&\le (t+2)^{-(2/q+3/8)}\|\mathbf{z}\|_X^2
\end{aligned}
$$

for every $t>0$ with some positive constant $C(q)$. Then, noting $p(2/q+3/8-3/4)>p/4>1$ by $3<q<16/5$ and (5.3), we have

$$
\|(\mathbf{u}\cdot\nabla)\mathbf{u}\|_{L^{3/4}_p((0,\infty),L_q(\mathbf{R}^3_-))} \le C(p,q)\|\mathbf{z}\|_X^2, \quad \|(\mathbf{u}\cdot\nabla)\mathbf{u}\|_{L^{2/q+3/8}_\infty((0,\infty),L_{q/2}(\mathbf{R}^3_-))} \le \|\mathbf{z}\|_X^2
$$

for a positive constant $C(p,q)$, which, combined with (5.6), furnishes that

$$
\|(\mathbf{u}\cdot\nabla)\mathbf{u}\|_{\mathbb{F}_1\cap\widetilde{\mathbb{F}}_1(3/4,2/q+3/8)} \le C(p,q)\|\mathbf{z}\|_X^2. \tag{5.7}
$$

Concerning $\partial_t H\partial_3\mathbf{u}$, we use Sobolev's inequality (cf. [2, Theorem 4.31]):

$$
\|f\|_{L_6(\mathbf{R}^3_-)} \le M_2\|\nabla f\|_{L_2(\mathbf{R}^3_-)} \tag{5.8}
$$

with a positive constant M_2. By (5.8), Hölder's inequality, and Sobolev's embedding theorem, we have for every $t>0$

$$
\begin{aligned}
&\|\partial_t H(t)\partial_3\mathbf{u}(t)\|_{L_q(\mathbf{R}^3_-)} \le \|\partial_t H(t)\|_{L_6(\mathbf{R}^3_-)}\|\nabla\mathbf{u}(t)\|_{L_r(\mathbf{R}^3_-)} \quad (1/6+1/r=1/q) \\
&\quad\le M_2\|\nabla\partial_t H(t)\|_{L_2(\mathbf{R}^3_-)}\|\nabla\mathbf{u}(t)\|_{W^1_q(\mathbf{R}^3_-)} \\
&\|\partial_t H(t)\partial_3\mathbf{u}(t)\|_{L_2(\mathbf{R}^3_-)} \le \|\partial_t H(t)\|_{L_6(\mathbf{R}^3_-)}\|\nabla\mathbf{u}(t)\|_{L_3(\mathbf{R}^3_-)} \\
&\quad\le M_2\|\nabla\partial_t H(t)\|_{L_2(\mathbf{R}^3_-)}\|\nabla\mathbf{u}(t)\|^a_{L_2(\mathbf{R}^3_-)}\|\nabla\mathbf{u}(t)\|^{1-a}_{L_q(\mathbf{R}^3_-)}, \\
&\|\partial_t H(t)\partial_3\mathbf{u}(t)\|_{L_{q/2}(\mathbf{R}^3_-)} \le \|\partial_t H(t)\|_{L_6(\mathbf{R}^3_-)}\|\nabla\mathbf{u}(t)\|_{L_s(\mathbf{R}^3_-)} \quad (1/6+1/s=2/q) \\
&\quad\le M_2\|\nabla\partial_t H(t)\|_{L_2(\mathbf{R}^3_-)}\|\nabla\mathbf{u}(t)\|^b_{L_2(\mathbf{R}^3_-)}\|\nabla\mathbf{u}(t)\|^{1-b}_{L_q(\mathbf{R}^3_-)},
\end{aligned} \tag{5.9}
$$

where we note that $0<a,b<1$ and

$$
3\left(\frac{1}{q}-\frac{1}{r}\right)=\frac{1}{2}<1, \quad \frac{1}{3}=\frac{a}{2}+\frac{1-a}{q}, \quad \frac{1}{s}=\frac{b}{2}+\frac{1-b}{q}. \tag{5.10}
$$

By (5.4) and (5.9), we obtain

$$
\begin{aligned}
&\|\partial_t H\partial_3\mathbf{u}\|_{L_p((0,\infty),L_q(\mathbf{R}^3_-))} \\
&\quad\le M_2\|\nabla\partial_t H\|_{L_\infty((0,\infty),L_2(\mathbf{R}^3_-))}\|\nabla\mathbf{u}\|_{L_p((0,\infty),W^1_q(\mathbf{R}^3_-))} \le M_2\|\mathbf{z}\|_X^2, \\
&\|\partial_t H\partial_3\mathbf{u}\|_{L_p((0,\infty),L_2(\mathbf{R}^3_-))} \\
&\quad\le M_2\|\nabla\partial_t H\|_{L_p((0,\infty),L_2(\mathbf{R}^3_-))}\|\nabla\mathbf{u}\|^a_{L_\infty((0,\infty),L_2(\mathbf{R}^3_-))}\|\nabla\mathbf{u}\|^{1-a}_{L_\infty((0,\infty),L_q(\mathbf{R}^3_-))} \\
&\quad\le M_1M_2\|\mathbf{z}\|_X^2.
\end{aligned} \tag{5.11}
$$

In addition, it follows from (5.9) that, for every $t>0$,

$$
\begin{aligned}
&\|\partial_t H(t)\partial_3\mathbf{u}(t)\|_{L_q(\mathbf{R}^3_-)} \le M_2(t+2)^{-m(q/2,2)-1/2}\|\mathbf{z}\|_X\\
&\quad\times\Big((t+2)^{-n(q/2,q)-1/8}\|\mathbf{z}\|_X+(t+2)^{-1/2}\{(t+2)^{1/2}\|\nabla^2\mathbf{u}(t)\|_{L_q(\mathbf{R}^3_-)}\}\Big),\\
&\|\partial_t H(t)\partial_3\mathbf{u}(t)\|_{L_{q/2}(\mathbf{R}^3_-)} \le M_2(t+2)^{-m(q/2,2)-1/2}\|\mathbf{z}\|_X\\
&\quad\times\Big((t+2)^{-n(q/2,2)-1/8}\|\mathbf{z}\|_X\Big)^b\Big((t+2)^{-n(q/2,q)-1/8}\|\mathbf{z}\|_X\Big)^{1-b}\\
&\quad= M_2(t+2)^{-(2/q+3/8)}\|\mathbf{z}\|_X^2,
\end{aligned}
$$

because $m(q/2,2)+1/2=2/q$ and, by (5.2) and (5.10),

$$
\begin{aligned}
bn\left(\frac{q}{2},2\right)+(1-b)n\left(\frac{q}{2},q\right)&=\frac{3b}{2}\left(\frac{2}{q}-\frac{1}{2}\right)+\frac{3(1-b)}{2q}\\
&=\frac{3}{2q}-\frac{3b}{2}\left(\frac{1}{2}+\frac{1}{q}-\frac{2}{q}\right)=\frac{3}{2q}-\frac{3}{2}\cdot\frac{6-q}{3(q-2)}\cdot\frac{q-2}{2q}=\frac{1}{4}.
\end{aligned}
$$

Since, by (5.2), (5.3), and $q<16/5<4$,

$$
\begin{aligned}
&p\left(m\left(\frac{q}{2},2\right)+\frac{1}{2}+n\left(\frac{q}{2},q\right)+\frac{1}{8}-\frac{3}{4}\right)=p\left(\frac{7}{2q}-\frac{5}{8}\right)>p\left(\frac{7}{8}-\frac{5}{8}\right)=\frac{p}{4}>1,\\
&m\left(\frac{q}{2},2\right)+\frac{1}{2}+\frac{1}{2}-\frac{3}{4}=\frac{2}{q}-\frac{1}{4}>\frac{2}{4}-\frac{1}{4}>0,
\end{aligned}
$$

we see that

$$
\begin{aligned}
&\|\partial_t H\partial_3\mathbf{u}\|_{L_p^{3/4}((0,\infty),L_q(\mathbf{R}^3_-))}\\
&\quad\le M_2\|\mathbf{z}\|_X\Big(\|(t+2)^{-(m(q/2,2)+1/2+n(q/2,q)+1/8-3/4)}\|_{L_p((0,\infty))}\|\mathbf{z}\|_X\\
&\quad+\|(t+2)^{-(m(q/2,2)+1/2+1/2-3/4)}\|_{L_\infty((0,\infty))}\|\nabla^2\mathbf{u}\|_{L_p^{1/2}((0,\infty),L_q(\mathbf{R}^3_-))}\Big)\\
&\quad\le C(p,q)\|\mathbf{z}\|_X^2,\\
&\|\partial_t H\partial_3\mathbf{u}\|_{L_\infty^{2/q+3/8}((0,\infty),L_{q/2}(\mathbf{R}^3_-))}\le M_2\|\mathbf{z}\|_X^2,
\end{aligned}
$$

which, combined with (5.11), furnishes that

$$
\|\partial_t H\partial_3\mathbf{u}\|_{\mathbb{F}_1\cap\widetilde{\mathbb{F}}_1(3/4,2/q+3/8)}\le C(p,q)\|\mathbf{z}\|_X^2.
$$

By (5.4), (5.7), and the last inequality, we have

$$
\begin{aligned}
&\|\mathbf{F}_1(\mathbf{u},H)\|_{\mathbb{F}_1\cap\widetilde{\mathbb{F}}_1(3/4,2/q+3/8)}\\
&\quad\le C(p,q)\left(\frac{\|\partial_t H\partial_3\mathbf{u}\|_{\mathbb{F}_1\cap\widetilde{\mathbb{F}}_1(3/4,2/q+3/8)}}{1-\|\nabla H\|_{L_\infty((0,\infty),L_\infty(\mathbf{R}^3_-))}}+\|(\mathbf{u}\cdot\nabla)\mathbf{u}\|_{\mathbb{F}_1\cap\widetilde{\mathbb{F}}_1(3/4,2/q+3/8)}\right)\\
&\quad\le C(p,q)\left(\frac{1}{1-\delta_0 M_1}+1\right)\|\mathbf{z}\|_X^2.
\end{aligned}
$$

In what follows, we suppose that δ_0 satisfies the condition: $\delta_0 M_1\le 1/2$, and we complete the required estimate of $\mathbf{F}_1(\mathbf{u},H)$ in (5.5) by the last inequality.

Next we consider $\mathbf{F}_2(\mathbf{u},H)$. By (5.4) it is clear that, for $r\in\{q,2\}$ and $j=1,2,3$,

$$
\|(\partial_t u_3\nabla H,\Delta u_3\nabla H,\mathcal{F}_{jj}(H)\mathbf{u},(\mathbf{u}\cdot\nabla H)\partial_3\mathbf{u})\|_{L_p(\mathbf{R}_+,L_r(\mathbf{R}^3_-))}\le C(p,q)\|\mathbf{z}\|_X^2 \tag{5.12}
$$

with some positive constant $C(p,q)$. In addition, it follows from Hölder's inequality and Sobolev's embedding theorem that, for every $t>0$ and $j=1,2,3$,

$$
\begin{aligned}
(5.13)\qquad & \|\partial_t u_3(t)\nabla H(t)\|_{L_q(\mathbf{R}^3_-)} \le \|\partial_t \mathbf{u}(t)\|_{L_q(\mathbf{R}^3_-)}\|\nabla H(t)\|_{L_\infty(\mathbf{R}^3_-)} \\
&\quad \le C(q)\|\partial_t \mathbf{u}(t)\|_{L_q(\mathbf{R}^3_-)}\|\nabla H(t)\|_{W^1_q(\mathbf{R}^3_-)} \\
&\quad \le C(q)(t+2)^{-m(q/2,q)-3/4}\|\mathbf{z}\|_X\{(t+2)^{1/2}\|\partial_t \mathbf{u}(t)\|_{L_q(\mathbf{R}^3_-)}\}, \\
& \|\Delta u_3(t)\nabla H(t)\|_{L_q(\mathbf{R}^3_-)} \le \|\nabla^2 \mathbf{u}(t)\|_{L_q(\mathbf{R}^3_-)}\|\nabla H(t)\|_{L_\infty(\mathbf{R}^3_-)} \\
&\quad \le C(q)\|\nabla^2 \mathbf{u}(t)\|_{L_q(\mathbf{R}^3_-)}\|\nabla H(t)\|_{W^1_q(\mathbf{R}^3_-)} \\
&\quad \le C(q)(t+2)^{-m(q/2,q)-3/4}\|\mathbf{z}\|_X\{(t+2)^{1/2}\|\nabla^2 \mathbf{u}(t)\|_{L_q(\mathbf{R}^3_-)}\}, \\
& \|\mathcal{F}_{jj}(H(t))\mathbf{u}(t)\|_{L_q(\mathbf{R}^3_-)} \\
&\quad \le C(q)\Big(\|\nabla \mathbf{u}(t)\|_{L_\infty(\mathbf{R}^3_-)}\|\nabla^2 H(t)\|_{L_q(\mathbf{R}^3_-)} + \|\nabla^2 \mathbf{u}(t)\|_{L_q(\mathbf{R}^3_-)}\|\nabla H(t)\|_{L_\infty(\mathbf{R}^3_-)}\Big) \\
&\quad \le C(q)\Big(\|\nabla \mathbf{u}(t)\|_{W^1_q(\mathbf{R}^3_-)}\|\nabla^2 H(t)\|_{L_q(\mathbf{R}^3_-)} + \|\nabla^2 \mathbf{u}(t)\|_{L_q(\mathbf{R}^3_-)}\|\nabla H(t)\|_{W^1_q(\mathbf{R}^3_-)}\Big) \\
&\quad \le C(q)(t+2)^{-m(q/2,q)-1/4}\|\mathbf{z}\|_X\Big((t+2)^{-n(q/2,q)-1/8}\|\mathbf{z}\|_X \\
&\quad + (t+2)^{-1/2}\{(t+2)^{1/2}\|\nabla^2 \mathbf{u}(t)\|_{L_q(\mathbf{R}^3_-)}\}\Big), \\
& \|(\mathbf{u}(t)\cdot\nabla H(t))\partial_3 \mathbf{u}(t)\|_{L_q(\mathbf{R}^3_-)} \le \|\mathbf{u}(t)\|_{L_\infty(\mathbf{R}^3_-)}\|\nabla H(t)\|_{L_\infty(\mathbf{R}^3_-)}\|\nabla \mathbf{u}(t)\|_{L_q(\mathbf{R}^3_-)} \\
&\quad \le C(q)\|\mathbf{u}(t)\|_{W^1_q(\mathbf{R}^3_-)}\|\nabla H(t)\|_{W^1_q(\mathbf{R}^3_-)}\|\nabla \mathbf{u}(t)\|_{L_q(\mathbf{R}^3_-)} \\
&\quad \le C(q)(t+2)^{-m(q/2,q)-m(q/2,q)-1/4-n(q/2,q)-1/8}\|\mathbf{z}\|_X^2
\end{aligned}
$$

with a positive constant $C(q)$. We thus obtain, for $j=1,2,3$,

$$
\begin{aligned}
(5.14)\qquad & \|\partial_t u_3 \nabla H\|_{L_p^{3/4}((0,\infty),L_q(\mathbf{R}^3_-))} \\
&\quad \le C(p,q)\|(t+2)^{-m(q/2,q)}\|_{L_\infty((0,\infty))}\|\mathbf{z}\|_X\|\partial_t \mathbf{u}\|_{L_p^{1/2}((0,\infty),L_q(\mathbf{R}^3_-))} \\
&\quad \le C(p,q)\|\mathbf{z}\|_X^2, \\
& \|\Delta u_3 \nabla H\|_{L_p^{3/4}((0,\infty),L_q(\mathbf{R}^3_-))} \\
&\quad \le C(p,q)\|(t+2)^{-m(q/2,q)}\|_{L_\infty((0,\infty))}\|\mathbf{z}\|_X\|\nabla^2 \mathbf{u}\|_{L_p^{1/2}((0,\infty),L_q(\mathbf{R}^3_-))} \\
&\quad \le C(p,q)\|\mathbf{z}\|_X^2, \\
& \|\mathcal{F}_{jj}(H)\mathbf{u}\|_{L_p^{3/4}((0,\infty),L_q(\mathbf{R}^3_-))} \\
&\quad \le C(p,q)\|\mathbf{z}\|_X\Big(\|(t+2)^{-(m(q/2,q)+1/4+n(q/2,q)+1/8-3/4)}\|_{L_p((0,\infty))}\|\mathbf{z}\|_X \\
&\quad + \|(t+2)^{-m(q/2,q)}\|_{L_\infty((0,\infty))}\|\nabla^2 \mathbf{u}\|_{L_p^{1/2}((0,\infty),L_q(\mathbf{R}^3_-))}\Big) \\
&\quad \le C(p,q)\|\mathbf{z}\|_X^2, \\
& \|(\mathbf{u}\cdot\nabla H)\partial_3 \mathbf{u}\|_{L_p^{3/4}((0,\infty),L_q(\mathbf{R}^3_-))} \\
&\quad \le C(p,q)\|(t+2)^{-(m(q/2,q)+m(q/2,q)+1/4+n(q/2,q)+1/8-3/4)}\|_{L_p((0,\infty))}\|\mathbf{z}\|_X^2 \\
&\quad \le C(p,q)\|\mathbf{z}\|_X^2
\end{aligned}
$$

with some positive constant $C(p,q)$, because, by (5.1), (5.3), and $3 < q < 16/5$,

$$p\left(m\left(\frac{q}{2},q\right)+m\left(\frac{q}{2},q\right)+\frac{1}{4}+n\left(\frac{q}{2},q\right)+\frac{1}{8}-\frac{3}{4}\right)$$
$$> p\left(m\left(\frac{q}{2},q\right)+\frac{1}{4}+n\left(\frac{q}{2},q\right)+\frac{1}{8}-\frac{3}{4}\right)=p\left(\frac{2}{q}-\frac{1}{8}\right)>\frac{p}{2}>1.$$

Analogously it holds that, for $j = 1,2,3$,

$$\|(\partial_t u_3\nabla H, \Delta u_3 \nabla H, \mathcal{F}_{jj}(H)\mathbf{u}, (\mathbf{u}\cdot\nabla H)\partial_3\mathbf{u})\|_{L^1_p((0,\infty),L_{q/2}(\mathbf{R}^3_-))} \le C(p,q)\|\mathbf{z}\|_X^2 \tag{5.15}$$

by the following inequalities and relations: For every $t > 0$ and $j = 1,2,3$,

$$\|\partial_t u_3(t)\nabla H(t)\|_{L_{q/2}(\mathbf{R}^3_-)}$$
$$\le C(q)(t+2)^{-m(q/2,q)-3/4}\|\mathbf{z}\|_X\{(t+2)^{1/2}\|\partial_t\mathbf{u}(t)\|_{L_q(\mathbf{R}^3_-)}\},$$
$$\|\Delta u_3(t)\nabla H(t)\|_{L_{q/2}(\mathbf{R}^3_-)}$$
$$\le C(q)(t+2)^{-m(q/2,q)-3/4}\|\mathbf{z}\|_X\{(t+2)^{1/2}\|\nabla^2\mathbf{u}(t)\|_{L_q(\mathbf{R}^3_-)}\},$$
$$\|\mathcal{F}_{jj}(H(t))\mathbf{u}(t)\|_{L_{q/2}(\mathbf{R}^3_-)}$$
$$\le C(q)(t+2)^{-m(q/2,q)-1/4}\|\mathbf{z}\|_X\Big((t+2)^{-n(q/2,q)-1/8}\|\mathbf{z}\|_X$$
$$+(t+2)^{-1/2}\{(t+2)^{1/2}\|\nabla^2\mathbf{u}(t)\|_{L_q(\mathbf{R}^3_-)}\}\Big),$$
$$\|(\mathbf{u}(t)\cdot\nabla H(t))\partial_3\mathbf{u}(t)\|_{L_{q/2}(\mathbf{R}^3_-)}$$
$$\le C(q)(t+2)^{-m(q/2,q)-m(q/2,q)-1/4-n(q/2,q)-1/8}\|\mathbf{z}\|_X^2,$$

which are obtained in the same manner as (5.13); $m(q/2,q)+3/4-1 = 1/(2q) > 0$; By (5.1), (5.3), and $3 < q < 16/5$,

$$p\left(m\left(\frac{q}{2},q\right)+m\left(\frac{q}{2},q\right)+\frac{1}{4}+n\left(\frac{q}{2},q\right)+\frac{1}{8}-1\right)$$
$$> p\left(m\left(\frac{q}{2},q\right)+\frac{1}{4}+n\left(\frac{q}{2},q\right)+\frac{1}{8}-1\right)=p\left(\frac{2}{q}-\frac{3}{8}\right)>\frac{p}{4}>1.$$

Thus, by (5.4), (5.12), (5.14), and (5.15), we obtain the required inequality of $\mathbf{F}_2(\mathbf{u},H)$ in (5.5).

Step 2 We set, for $\mathbf{z} = (\mathbf{u},\theta,h,H) \in X_{\delta_0}$,

$$\mathbf{f}_1 = \mathbf{F}_1(\mathbf{u},H),\quad \mathbf{f}_2 = \mathbf{F}_2(\mathbf{u},H),\quad g_h = G_h(\mathbf{u},H),$$
$$\mathbf{f}_d = \mathbf{F}_d(\mathbf{u},H),\quad f_d = F_d(\mathbf{u},H),\quad \mathbf{g} = \mathbf{G}(\mathbf{u},H)$$

in (4.1), and we denote the solution of (4.1) with the initial data $(\mathbf{u}_0, h_0)$ by $\Phi(\mathbf{z})$.

By (5.5) and Proposition 4.1, we have

$$\|\Phi(\mathbf{z})\|_X \le M\left(\|(\mathbf{u}_0,h_0)\|_{\mathbb{I}_1\times\mathbb{I}_2}+\|\mathbf{z}\|_X^2\right) \le M\left(\varepsilon_0+\delta_0^2\right)$$

with some positive constant M. We here choose positive numbers ε_0, δ_0 satisfying $M\delta_0 \le 1/2$ and $M\varepsilon_0 \le \delta_0/2$, and then Φ is a mapping from X_{δ_0} to itself. We similarly have, for $\mathbf{z}_1,\mathbf{z}_2 \in X_{\delta_0}$,

$$\|\Phi(\mathbf{z}_1)-\Phi(\mathbf{z}_2)\|_X \le M\delta_0\|\mathbf{z}_1-\mathbf{z}_2\|_X \le \frac{1}{2}\|\mathbf{z}_1-\mathbf{z}_2\|_X$$

by taking a smaller $\delta_0 > 0$ if necessary.

We thus see that Φ is a contraction mapping on X_{δ_0}, so that Φ has a unique fixed point $\mathbf{z}^* = (\mathbf{u}^*, \theta^*, h^*, H^*) \in X_{\delta_0}$ by the contraction mapping theorem. The $\mathbf{z}^*$ is a unique solution to (1.6) and (1.3). This completes the proof of theorem.

References

[1] H. Abels. The initial-value problem for the Navier-Stokes equations with a free surface in L_q-Sobolev spaces. *Adv. Differential Equations*, 10(1):45–64, 2005.

[2] R. A. Adams and J. J. F. Fournier. *Sobolev spaces*, volume 140 of *Pure and Applied Mathematics*. Elsevier/Academic Press, Amsterdam, 2nd edition, 2003.

[3] G. Allain. Small-time existence for the Navier-Stokes equations with a free surface. *Appl. Math. Optim.*, 16(1):37–50, 1987.

[4] H. Bae. Solvability of the free boundary value problem of the Navier-Stokes equations. *Discrete Contin. Dyn. Syst.*, 29(3):769–801, 2011.

[5] J. T. Beale. The initial value problem for the Navier-Stokes equations with a free surface. *Comm. Pure Appl. Math.*, 34(3):359–392, 1981.

[6] J. T. Beale. Large-time regularity of viscous surface waves. *Arch. Rational Mech. Anal.*, 84(4):307–352, 1983/84.

[7] J. T. Beale and T. Nishida. Large-time behavior of viscous surface waves. In *Recent Topics in Nonlinear PDE II*, volume 128 of *North-Holland Math. Stud.*, pages 1–14. North-Holland, Amsterdam, 1985.

[8] J. Bergh and J. Löfström. *Interpolation spaces: An introduction*, volume 223 of *Grundlehren der Mathematischen Wissenschaften*. Springer-Verlag, Berlin; Heidelberg; New York, 1976.

[9] Y. Hataya. A remark on Beale-Nishida's paper. *Bull. Inst. Math. Acad. Sin. (N. S.)*, 6(3):293–303, 2011.

[10] Y. Hataya and S. Kawashima. Decaying solution of the Navier-Stokes flow of infinite volume without surface tension. *Nonlinear Anal.*, 71(12):e2535–e2539, 2009.

[11] J. Prüss and G. Simonett. On the Rayleigh-Taylor instability for the two-phase Navier-Stokes equations. *Indiana Univ. Math. J.*, 59(6):1853–1871, 2010.

[12] J. Prüss and G. Simonett. On the two-phase Navier-Stokes equations with surface tension. *Interfaces Free Bound.*, 12(3):311–345, 2010.

[13] J. Prüss and G. Simonett. Analytic solutions for the two-phase Navier-Stokes equations with surface tension and gravity. In *Parabolic Problems*, volume 80 of *Progr. Nonlinear Differential Equations Appl.*, pages 507–540. Birkhäuser/Springer Basel AG, Basel, 2011.

[14] H. Saito. *Free boundary problems of the incmopressible Navier-Stokes equations in some unbounded domains.* PhD thesis, Waseda University, 2015.

[15] H. Saito. On the $\mathcal{R}$-boundedness of solution operator families of the generalized stokes resolvent problem in an infinite layer. *Math. Methods Appl. Sci.*, 2015. to appear.

[16] H. Saito and Y. Shibata. On decay properties of solutions to the Stokes equations with surface tension and gravity in the half space. *J. Math. Soc. Japan*, 2015. to appear.

[17] Y. Shibata. On some free boundary problem of the Navier-Stokes equaions in the maximal L_p-L_q regularity class. *arXiv:1501.02054*, 2015. http://arxiv.org/abs/1501.02054

[18] Y. Shibata and S. Shimizu. On the maximal L_p-L_q regularity of the Stokes problem with first order boundary condition; model problems. *J. Math. Soc. Japan*, 64(2):561–626, 2012.

[19] A. Tani. Small-time existence for the three-dimensional Navier-Stokes equations for an incompressible fluid with a free surface. *Arch. Rational Mech. Anal.*, 133(4):299–331, 1996.

[20] A. Tani and N. Tanaka. Large-time existence of surface waves in incompressible viscous fluids with or without surface tension. *Arch. Rational Mech. Anal.*, 130(4):303–314, 1995.

[21] H. Triebel. *Theory of function spaces*, volume 78 of *Monographs in Mathematics*. Birkhüser Verlag, Basel, 1st edition, 1983.

Department of Pure and Applied Mathematics, Graduate School of Fundamental Science and Engineering, Waseda University, 3-4-1 Ookubo, Shinjuku-ku, Tokyo 169-8555, Japan

E-mail address: `hsaito@aoni.waseda.jp`

Department of Mathematics and Research Institute of Science and Engineering, Waseda University, 3-4-1 Ookubo, Shinjuku-ku, Tokyo, 169-8555, Japan

E-mail address: `yshibata@waseda.jp`